SECTION DE MATHÉMATIQUES

DR. MATHIAS BRAUN & WENHAO ZHAO

FUNCTIONAL ANALYSIS 2
SPRING 2025
SHEET 4

Exercises for 'Functional Analysis 2' [MATH-404]

(10/03/2025)

Ex 4.1 (Linear maps on L^p with $0 < p < 1$)
Let $p \in (0,1)$; let L^p denote the space of Lebesgue measurable functions on $\mathbb{R}$ for which

$$\rho(f) = \int_{\mathbb{R}} |f(x)|^p \, dx < +\infty,$$

endowed with the topology induced by the metric $d(f,g) = \rho(f-g)$.

a) Show that the only convex and open subsets of L^p are $\emptyset$ and L^p itself.
Hint: Since L^p is a TVS, wlog the open set contains the origin. Given $r > 0$ and $f \in L^p$, write $f = \sum_{i=0}^n \lambda_i g_i$ with $\lambda_i \in (0,1)$ and $\sum_i \lambda_i = 1$ and functions $g_i = \lambda_i^{-1} f \chi_{I_i}$, where the intervals I_i form a partition of $\mathbb{R}$ such that $\rho(g_i) < r$ for each i.

b) Let $T \colon L^p \to X$, where X is a LCTVS, be a continuous linear mapping. Prove that

$$Tf = 0 \quad \text{for all } f \in L^p.$$

c) Deduce that $(L^p)' = \{0\}$.

Ex 4.2 (The spaces $C(\Omega)$)
Let Ω be a nonempty open subset of $\mathbb{R}^d$.

a) Show that there exists a sequence of compact subsets $(K_n)_{n\in\mathbb{N}}$ such that

$$K_n \subset \operatorname{int}(K_{n+1}) \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} \operatorname{int}(K_n) = \Omega, \tag{$\star$}$$

where $\operatorname{int}(K)$ denotes the **interior** of a set K, i.e. the largest open set contained in K.
Hint: If $\Omega \neq \mathbb{R}^d$, work with the distance function to the closed set $\mathbb{R}^d \setminus \Omega$.

Let $C(\Omega)$ be the vector space of all continuous $f \colon \Omega \to \mathbb{R}$ and consider the family of seminorms

$$p_n(f) := \max_{x\in K_n} |f(x)|,$$

where $(K_n)_{n\in\mathbb{N}}$ is any sequence of compact sets satisfying $(\star)$.

b) Show that $C(\Omega)$ with this family of seminorms is a LCTVS whose topology does not depend on the choice of a sequence $(K_n)_{n\in\mathbb{N}}$. Give an example of a translation invariant metric on $C(\Omega)$ and demonstrate that $C(\Omega)$ is a Fréchet space. Is it normable?

c) Give an example of a bounded and closed set $E \subset C(\Omega)$ which is not compact.

Ex 4.3 (Continuous functionals)
For each of the following LCTVS X (endowed with the topology defined in the previous exercises and/or the lecture notes), show that the linear functional Λ on X as defined below is continuous :

a) $X = C(\Omega)$, $\Lambda_{x_0}(f) := f(x_0)$, where $x_0 \in \Omega$;

b) $X = C(\Omega)$, $\Lambda_g(f) := \int_\Omega f(x)g(x)\,dx$, where g is continuous with compact support in Ω ;

c) $X = \mathcal{D}_{[a,b]}$, $\Lambda_{x_0}^{(k)}(f) := f^{(k)}(x_0)$, where $k \in \mathbb{N}_0$ and $x_0 \in [a, b]$.

Ex 4.4 (Locally compact Hausdorff-TVS are finite dimensional*)
Let X be a Hausdorff topological vector space such that 0 has an open neighborhood U with $\overline{U}$ being compact. Show that X is finite dimensional. You may follow the guideline below :

a) Show that there exist $x_1, \ldots, x_n \in X$ such that $\overline{U} \subset \bigcup_{i=1}^n (x_i + \frac{1}{2}U)$.

b) Define $Y = \text{span}(x_1, \ldots, x_n)$ and deduce that Y is closed. Moreover, show that $\frac{1}{2}U \subset Y + \frac{1}{4}U$.

c) Prove by induction that

$$U \subset \bigcap_{n=1}^{+\infty} (Y + 2^{-n}U).$$

d) Deduce that $U \subset Y$ and finally $X \subset Y$ to conclude.